A class of integral operators on the Dirichlet space of the upper half plane

BY

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DECLARATION

This thesis is my own work and has not been presented for a degree award in any other institution.

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This thesis has been submitted for examination with our approval as the university supervisors.

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Dedication

To my parents Mr. & Mrs. Zakaria Njuguna.

ABSTRACT

Exploring integral operators and spaces of analytic functions remains a focal point for many mathematicians. While majority of the research has centered around the analytic spaces of the unit disk. There has been a growing interest in extending these studies to analytic spaces of the upper half plane. The properties of integral operators on Hardy and Bergman spaces of the upper half plane have largely been determined. However, there is still more to explore regarding the Dirichlet space of the upper half-plane. In this study, we have focused on the properties of integral operators on Dirichlet spaces of the upper half plane $\mathcal{D}(\mathbb{U})$. Specifically, we have determined the semigroup properties of composition semigroup; constructed a class on integral operators and investigated its properties on the Dirichlet space of the upper half plane. Applying methods similar to those utilized in the related works of Agwang' and Oyugi, we identified the semigroup properties. We have constructed a class of integral operators which is acting on the Dirichlet space of the upper half plane employing the approach of strongly continuous semigroup of composition operators on Banach spaces. By employing the spectral mapping theorems and Hille-Yosida theorem we successfully identified the spectra and the norm properties of integral operators. The findings of this research contributes meaningfully to the existing literature, providing valuable insights for pure mathematicians advancing in this area.

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Chapter 1

Introduction

1.1 Background of the study

Since 1992, studies have been conducted on integral operators in spaces of analytical functions. Rochberg and Wu [17], in their study considered Toeplitz operators on Dirichlet space of the unit disc in the complex plane, $\mathbb{C}$. Aleman and Siskakis [3] investigated an integral operators on Hardy spaces. They researched also on integral operators on Hardy spaces [4]. More recently, attention has turned towards integral operators on analytic spaces of the upper half plane, $\mathbb{U}$. In 2013 Arvanitidis and Siskakis [6] investigated Cesáro operators on the Hardy spaces of the upper half plane, $H^p(\mathbb{U})$. A similar approach was taken by Ballamoole et al [7] , they investigated Cesáro like operator on both $H^p(\mathbb{U})$ and $L^p_a(\mathbb{U})$, the Bergman spaces of the upper half plane. Agwang' [1] considered groups of composition operators on the Dirichlet spaces of the upper half plane $\mathcal{D}(\mathbb{U})$. He established the strong continuity of the induced groups of composition operators, along with the existence of their infinitesimal generator Γ and domain. Oyugi [15] constructed a Cesáro-type integral operator acting on $\mathcal{D}(\mathbb{U})$. However, a general class of integral operators on the Dirichlet spaces of the upper half plane has never been obtained. In this study we have considered the entire class of integral operators on the Dirichlet spaces of the upper half plane induced by the scaling class and providing the analysis for the translation class.

1.2 Basic concepts

1.2.1 Normed spaces

Let X be a vector space over a scalar field $\mathbb{F}$. A function $\|.\| : X \to \mathbb{F}$ is a **norm** if the following conditions holds.

1. $\|x\| = 0 \Leftrightarrow x = 0,$
2. $\|\lambda x\| = |\lambda|\|x\|$ for every $x \in X$ and $\lambda \in \mathbb{F}$,
3. $\|x + y\| \leq \|x\| + \|y\|$ for every $x, y \in X$.

Let X be a vector space and $\|.\|$ be a norm on X, then the pair $(X, \|.\|)$ is called a **normed space**.

Let x_n denote a sequence in a normed space X. Then, (x_n) is **convergent** to a limit, say, x if $\forall \epsilon > 0 \, \exists N \geq 0$ such that $\|x_n - x\| < \epsilon \, \forall n \geq N$. In this case, we write $x_n \to x$ as $n \to \infty$. (x_n) is said to be **Cauchy** in X if $\forall \epsilon \geq 0 \, \exists \, N = N(\epsilon)$ such that $\forall m, n \geq N$, $\|x_n - x_m\| < \epsilon$.

A normed space X is said to be **complete** if every Cauchy sequence in X converges to an element of X. A complete normed space is called a

Banach space.

A **Hilbert space** is a Banach space $(X, \|.\|)$ whose norm is given by their inner product via relation $\|x\|^2 \langle x, x \rangle, \forall x \in X$, where $\langle ., . \rangle$ denotes the inner product. A complete inner product space is called Hilbert space, H.

1.2.2 The unit Disk $(\mathbb{D})$ and the upper half plane $(\mathbb{U})$

Let the set of all complex numbers be denoted by $\mathbb{C}$. The open unit disk $\mathbb{D}$ of the complex plane $\mathbb{C}$ is defined as $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$. Let the area measure dA on the unit disk be normalized such that the area of $\mathbb{D}$ is 1. Then $dA = \frac{1}{\pi}dxdy = \frac{r}{\pi}drd\theta$, where $re^{i\theta} = x + iy = z \in \mathbb{D}$. The upper half plane $\mathbb{U}$ of the complex plane is set $\mathbb{U} = \{w \in \mathbb{C} : \Im(w) > 0\}$, where $\Im(w)$ represents the imaginary part of $w \in \mathbb{C}$. Additionally, the area Lebesgue measure on $\mathbb{U}$ is denoted by $dA(w)$. The function $\psi : \mathbb{U} \to \mathbb{U}$ given by $\psi(z) = \frac{1(1+z)}{1-z}$ is conformal and its inverse $\psi^{-1} : \mathbb{U} \to \mathbb{D}$ is given by $\psi^{-1} = \frac{w-1}{w+1}$. See [14, 23] for details.

1.2.3 Analytic function

Let an open subset of $\mathbb{C}$ be represented by Ω, consider a function f that maps Ω onto the complex plane. The function f is said to be **complex differentiable** at $z_0 \in \Omega$ if the limit

$$f'(z_0) =^{lim}_{\Delta x \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

exists. Also, f is analytic or holomorphic on Ω if it is differentiable within a neighborhood of every point in Ω. f is said to be an **entire function** if it is analytic on $\mathbb{C}$, that is $\Omega = \mathbb{C}$.
A bijective and holomorphic map is known as **biholomorphic**. A biholomorphic map of an open subset of $\mathbb{C}$ into itself is called an **automorphism**. The automorphism set on Ω, denoted by $Aut(\Omega)$, constitutes a group, where the group operation is given by composition of maps. If a function φ maps Ω into itself, then φ is called a **self map.** It is important to note that self analytic maps of Ω essentially represents the automorphism of Ω. See [15, 16, 23] for details.

1.2.4 Spaces of analytic function

Let $\mathcal{H}(\Omega)$ denote the set of all analytic function f that maps Ω onto $\mathbb{C}$. Then:

1. $H^p(\mathbb{D})$ $1 \leq p < \infty$, is the set of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{H^p(\mathbb{D})} =^{sup}_{0<r<1} \frac{1}{2\pi} \left(\int_{-\pi}^{\pi} |f(re^{i\theta})|^p d\theta \right)^{\frac{1}{p}} < \infty.$$

Similarly $H^p(\mathbb{U})$, $1 \leq p < \infty$ consists of all $f \in \mathcal{H}(\mathbb{U})$ for which

$$\|f\|_{H^p(\mathbb{U})} =^{\sup}_{y>0} \left(\int_{-\infty}^{\infty} |f(x+iy)|^p dx \right)^{\frac{1}{p}} < \infty.$$

$H^p(.)$ are complete normed spaces with $\|.\|_{H^p(.)}$ defined as the norm. In particular, $H^2(.)$ are Hilbert spaces.

2. $L_a^p(\mathbb{D})$, $1 \le p < \infty$ is defined as set of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$L_a^p(\mathbb{D}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L_a^p(\mathbb{D})} := \left(\int_{\mathbb{D}} |f(z)|^p dA(z) \right)^{\frac{1}{p}} < \infty \right\}.$$

Similarly, $L_a^p(\mathbb{U})$, $1 \le p < \infty$, consists of all $f \in \mathcal{H}(\mathbb{U})$ for which

$$L_a^p(\mathbb{U}) := \left\{ f \in \mathcal{H}(\mathbb{D}) : \|f\|_{L_a^p(\mathbb{U})} := \left(\int_{\mathbb{U}} |f(w)|^p dA(w) \right)^{\frac{1}{p}} \right\}.$$

L_a^p are Banach spaces under the norm $\|.\|_{L_a^p(.)}$ additionally L_a^2 is a Hilbert space.

3. $\mathcal{D}(\mathbb{D})$, $1 \le p < \infty$ is defined as a set of all $f \in \mathcal{H}(\mathbb{D})$ for which

$$\|f\|_{D_1(\mathbb{D})}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$$

with $\|f\|_{\mathcal{D}(\mathbb{D})}^2 = |f(0)|^2 + \|f\|_{D_1(\mathbb{D})}^2$ and the seminorm as $\|.\|_{\mathcal{D}_1(\mathbb{D})}$ on $\mathcal{D}(\mathbb{D})$.

Similarly, $\mathcal{D}(\mathbb{U})$, consists of all $f \in \mathcal{H}(\mathbb{U})$ for which

$$\|f\|_{D_1(\mathbb{U})}^2 = \int_{\mathbb{D}} |f'(z)|^2 dA(z) < \infty$$

with the norm as $\|f\|_{\mathcal{D}(\mathbb{U})}^2 = |f(i)|^2 + \|f\|_{\mathcal{D}_1(\mathbb{U})}^2$ and the seminorm as $\|.\|_{\mathcal{D}_1(\mathbb{U})}$ on $\mathcal{D}(\mathbb{U})$.

Considering the norm $\|.\|_{\mathcal{D}(.)}$, $\mathcal{D}(.)$ is a complete normed space and also complete with respect to inner product defined on $\mathbb{D}$ by

$$\langle f, h \rangle_{\mathcal{D}(\mathbb{D})} = \langle f(0), h(0) \rangle + \int_{\mathbb{D}} f'(z)\overline{h'(z)} dA(z) \quad \forall f, h \in \mathcal{D}(\mathbb{D})$$

and for $\mathbb{U}$ by

$$\langle f,h\rangle_{\mathcal{D}(\mathbb{U})} = \langle f(i),h(i)\rangle + \int_{\mathbb{U}} f'(w)\overline{h'(w)}dA(w) \quad \forall f,h \in \mathcal{D}(\mathbb{U}).$$

We refer [9, 13, 21, 23] for a comprehensive theory.

1.2.5 Operators and functions

Consider Banach spaces X and Y over the field $\mathbb{F}$. An operator $T : X \to Y$ is linear if $\forall x,y \in X, \alpha,\beta \in \mathbb{F}$ we have $T(\alpha x + \beta y) = \alpha Tx + \beta Ty$. An operator T defined on Banach space X is considered **closed** if, whenever a sequence (x_n) in the domain of T converges to some $x \in X$ and Tx_n converges to some $y \in X$, then x is the domain of T and $Tx = y$. If $T : X \to Y$ is closed, we define the spectrum of $T, \sigma(T)$, by

$$\sigma(T) = \{\lambda \in \mathbb{C} : \lambda I - T \quad \text{is not invertible}\}.$$

The compliment of spectrum $\mathbb{C}/\rho(T)$ is denoted by $\rho(T)$ and is known as the **resolvent set** of T. The **point spectrum** of an operator T is the set of all eigenvalues of T, it is denoted by $\sigma_p(T)$ and the **spectral radius** of T is denoted by $r(T)$ is

$$r(T) =^{\sup}_{\lambda\in\sigma(T)} |\lambda|$$

The operator $R(\lambda, T) := (\lambda I - T)^{-1}$ mapping X into X is the resolvent T at $\lambda \in \rho(T)$.

A linear operators $T : X \to Y$ is said to be bounded if there exists a constant $M > 0$ such that,

$$\|Tx\| \leq M\|x\|, \ \forall x \in X$$

in such cases, we define the **operator norm** of T, $\|T\|$ by

$$\|T\| = \sup\{\|Tx\| : x \in X, \|x\| \leq 1\}$$

The set of all bounded linear operators from X into Y is denoted as $L(X,Y)$, and is written as $L(X) = L(X,X)$.
The **range** of a linear operator $T : X \to Y$ is the set,

$$R(T) = \{y \in Y : y = Tx, \text{ for some } x \in X\}.$$

The null space of $T : X \to Y$ is the set,

$$N(T) = \{x \in X : Tx = 0\}.$$

If $T \in L(X,Y)$, then $N(T)$ is a closed linear subspace of X, while $R(T)$ is a linear subspace of Y which is not necessary closed.
A **functional** is an operator whose range lies in $\mathbb{F}$. We shall denote the set of all bounded linear functions $L(X,\mathbb{F})$ by X^*, the **dual space** of X.
The operator T is said to be an **isometry** if it preserves norm, that is,

$$\|Tx\| = \|x\| \quad \forall x \in X.$$

In an inner product space X, the definition becomes

$$\langle x, x \rangle = \langle Tx, Tx \rangle \ \ \forall x \in X.$$

Equivalently $T^*T = I$, given that T^* denotes the **adjoint** of T. This also means that the isometries preserve inner products as

$$\langle Tu, Tv \rangle = \langle u, T^*Tv \rangle = \langle u, v \rangle .$$

See [15, 18] for more details.

1.2.6 Some integral operators

Let f be a measurable function on $\mathbb{R}$ and k be a fixed measurable function on $\mathbb{R}^2$. The integral operator L_k with kernel K is formally defined by

$$L_k F(x) = \int_{-\infty}^{\infty} k(x,y) f(y) dy.$$

Example 1.1

Voltera type integral operators- This is a bounded linear operator on the space of holomorphic function Ω. This operator is induced by a holomorphic symbol $g : \Omega \to \mathbb{C}$ defined by

$$V_g f(z) = \int_0^z f(\zeta) g'(\zeta) d\zeta$$

for $z \in \Omega$ and $f \in \Omega$. For more details see [15].

1.2.7 Semigroup of linear operators

A one parameter family $(T_t)_{t\geq 0}$ is a semigroup bounded linear operator on Banach space X if it satisfies;

(a) $T_0 = I$, where I is the identity operator on X,

(b) $T_{t+s} = T_t \circ T_s \quad \forall t, s \geq 0$. (Semigroup property)

$(T_t)_{t\in\mathbb{R}}$ is a group on X if both $(T_t)_{t\geq 0}$ and $(T_t)_{t\leq 0}$ are semigroups on X. The semi group, $(T_t)_{t\geq 0} \subseteq L(X)$ is **strongly continuous** if $\forall x \in X$, $\lim\limits_{t\to 0^+} \|T_t x - x\| = 0$.
$\mathcal{C}_0$ - semigroup is called a strongly continuous semigroup. The **infinitesimal generator** Γ of $(T_t)_{t\geq 0}$ is given by

$$\Gamma x = \lim_{t\to 0^+} \frac{T_t x - x}{t} = \frac{\partial}{\partial t})(T_t x)\bigg|_{t=0}$$

for each $x \in$ dom (Γ), where the domain is given by

$$\text{dom}(\Gamma) = \left\{ x \in X : \lim_{t\to 0^+} \frac{T_t x - x}{t} \quad \text{exists} \right\}.$$

For details see [14, 16, 20].

1.2.8 Composition operator and semigroups

Let φ be an analytic function on an open subset Ω of $\mathbb{C}$, then φ is a self analytic map that is $\varphi : \Omega \to \Omega$. A family $(\varphi_t)_{t\geq 0}$ of analytic self maps on Ω is a semigroup if it satisfies the following conditions:

(i) $\varphi_0(z) = z$, the identity map of Ω.

(ii) $\varphi_{t+s} = \varphi_t \circ \varphi_s \quad \forall t, s \geq 0$.

$(\varphi_t)_{t\in\mathbb{R}}$ is a group of both $(\varphi_t)_{t\geq 0}$ and $(\varphi_t)_{t\leq 0}$ are semigroups.
The **composition operator** $\mathcal{C}_\varphi$ induced by φ and acting on $\mathcal{H}(\Omega)$ is defined by

$$\mathcal{C}_\varphi g = g \circ \varphi \qquad \forall g \in \mathcal{H}(\Omega).$$

On the other hand the composition semigroup

$$\mathcal{C}_{\varphi_t}(g) = g \circ \varphi_t \qquad \forall g \in \mathcal{H}(\Omega)$$

For more details see [15, 20, 21]

1.3 Statement of the problem

The study of integral operators and their properties have majorly focused on analytic spaces of the unit disk. Such spaces includes Hardy spaces, Bergman spaces and Dirichlet spaces. However, the corresponding study on analytic space of the upper half plane, $\mathbb{U}$ is much less complete. Recently the properties of integral operator on Hardy and Bergman spaces of the upper half plane has been determined. In this study therefore, we completed the investigation on the composition semigroup of the scaling class which was partially done. We have also constructed a class of integral operators on $\mathcal{D}(\mathbb{U})$ induced by the translation class as the resolvents of the infinitesimal generator Γ and investigated its properties.

1.4 Objectives of the study

The main objective of the study was to determine the properties of a class of integral operators on the Dirichlet space of the upper half plane. The specific objectives were to:

1. Determine the semigroup properties of a composition semigroup on the Dirichlet space of the upper half plane.
2. Construct a class of integral operators on $\mathcal{D}(\mathbb{U})$.
3. Investigate properties of the class of integral operators constructed in (2) above.

1.5 Significance of the study

Integral operators on spaces of analytic functions on the unit disk $\mathbb{D}$ have been investigated. Not so much has been done on the upper half plane $\mathbb{U}$. Findings from this study have added new information to the field of pure mathematics thereby providing valuable insights to the current available literature and help in advancing of research on this and other related areas. Besides, the outcomes of this research holds potential utility in the exploration of integral equations and operators.

1.6 Research methodology

To determine the semigroup properties of a composition semigroup on $\mathcal{D}(\mathbb{U})$, we proved that this composition semigroup induced by both scaling and translation classes forms a group, it is a linear isometry and strongly continuous on $\mathcal{D}(\mathbb{U})$. We also determined the infinitesimal generator and its domain.

To construct a class of integral operators on the $\mathcal{D}(\mathbb{U})$, we have used Laplace transform which gives a class of integral operators for both scaling and translation classes.

Investigating the properties of the class of integral operators constructed, we have used spectral mapping theorems for strong continuous groups and for resolvent to determine the spectrum of the integral operators.

To obtain the norm properties of the constructed operators as well as the spectral radius, we applied the Hille-Yosida theorem.

Chapter 2

Literature review

Studies on integral operators on analytic spaces of the unit disk has been done. It was introduced in 1992 by Rochberg and Wu [17] who considered Toeplitz operators on $\mathcal{D}(\mathbb{D})$ in $\mathbb{C}$, for the operator to be bounded and compact on this space, they obtained the necessary and sufficient conditions on the symbols of the operator.

In 1995, Aleman and Siskakis [3] determined an integral operator on H^p, they studied the operator T and characterized the function g for which the operator T_g is bounded or compact on H^p or Schatten classes on H^p. If $g(z) = z$ then T is integral operator on H^p while if $g(z) = \log(1/(1-z))$, then T is the Cesáro operator.

Semigroup operators on Dirichlet spaces were studied in 1996 by Siskakis [20]. He showed that semigroup of composition operators are strongly continuous on the Dirichlet space. He further showed how infinitesimal generator is identified and gave some spectral properties.

In 1997, Aleman and Siskakis [4] investigated operators of the form $T_g(f)(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta$, on weighted Bergman spaces $L_a^2(\omega)$ of the unit disk in their paper, Integration operators on Bergman spaces.

Liankuo [22], in 2009 studied Hankel operator on the $\mathcal{D}(\mathbb{D})$ and proved its relation on Hankel operator on L_a^2 and H^2. Voltera operator on $H^p(\mathbb{D})$ was studied by Brevig et al [9].

In recent times, there has been a growing interest in studying classes of integral operators on $\mathbb{U}$. In 2013, Arvanitidis and Siskakis [6] determined a Cesáro operator on Hardy spaces of the upper half plane and identified resolvents for appropriate semigroups of composition operators. They found the norm and the spectrum in each case.

Cesáro-like operator on $H^p(\mathbb{U})$ and $L_a^p(\mathbb{U})$ was studied in 2016 by Ballamoole et al [7]. They developed integral operators associated with strongly continuous groups of invertible isometries on the H^p and the weighted Bergman spaces of $\mathbb{U}$. They specifically obtained the spectrum and point spectrum of the generator and represented resolvents as integral operators related to the Cesáro operator investigated by Arvanitidis and Siskakis on $H^p(\mathbb{U})$.

In 2020, Agwang' [1] investigated groups of composition operators on $\mathcal{D}(\mathbb{U})$. He established composition groups induced by the scaling, rotation and translation groups. He also delved into semigroups and spectral characteristics of every group on $\mathcal{D}(\mathbb{U})$. He demonstrated that the induced groups of composition operators exhibit strong continuity, and both their Γ and domain are existent.

In 2021, integral operators of Cesáro-type that acts on $\mathcal{D}(\mathbb{U})$ were constructed by Oyugi [15]. However, she did not obtain the general class of integral operators.

Therefore, in this study, we will examine the characteristics of integral operators within the Dirichlet space of the upper half plane induced by scaling and translation classes. The following results shall be useful:

Theorem 2.1 (Cauchy - Schwarz inequality)

In an inner product space M over $\mathbb{C}$. Then for all $x, y \in M$,

$$|\langle x, y\rangle| \leq \|x\|\|y\|.$$

Theorem 2.2 (Laplace Transform)

Suppose X is a Banach space, and let Γ be the infinitesimal generator of a strong continuous semigroup of contractions $(T_t)_{t\geq 0} \subseteq L(X)$ for any $\lambda > 0$ and $h \in X$ $R(\lambda, \Gamma)$ is defined by

$$R(\lambda, \Gamma)h(z) = \int_0^\infty e^{\lambda t} T_t h(z) dt,$$

and this integral converges in the norm.

Theorem 2.3 (Spectrum of invertible and non-invertible isometry)

Let X be a Banach space and $T : X \to X$ on a Banach space X. If T is an invertible isometry, then

$$\partial\mathbb{D} \supseteq \sigma(T),$$

where $\partial\mathbb{D}$ is the unit circle's boundary. If T is a non-invertible isometry, then

$$\partial\mathbb{D} = \sigma(T).$$

Theorem 2.4 (Hille-Yoside)

Let $(T_t)_{t\geq 0}$ be a one-parameter semigroup of linear operators on a Banach space X and Γ be infinitesimal generator. The Hille-Yosida theorem states that if:

(i) *Γ is closed and the domain of Γ is dense in X.*

(ii) The resolvent $R(\lambda, \Gamma)$ *exists for some* $\lambda \geq 0$.

(iii) $\|R(\lambda, \Gamma)\| \leq \frac{1}{\lambda}$.

Then, the operator Γ *generates a bounded semigroup of operators.*

Theorem 2.5 (Spectral mapping theorem for resolvents)

Let X *be a Banach space and* $T : X \to X$ *be closed, and* $\lambda \in \rho(T)$. *Then;*

1. $\sigma(R(\lambda, T)) \setminus \{0\} = (\lambda - \sigma(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(T)\right\},$
2. $\sigma_p(R(\lambda, T)) \setminus \{0\} = (\lambda - \sigma_p(T))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma_p(T)\right\}.$

Theorem 2.6 (Spectral mapping theorem for semigroup)

Consider a strongly continuous semigroup $(T_t)_{t\geq 0}$ *and its infinitesimal generator,* Γ. *Then the spectrum;*

$$e^{t\sigma(\Gamma)} \subset \sigma(T_t)$$

and for this spectrum

$$\sigma_p(T_t) = e^{t\sigma_p(\Gamma)}.$$

Theorem 2.7 (Closed graph theorem)

Suppose X *and* Y *are Banach spaces, and let* $T : X \to Y$ *be linear operator. Then* T *is closed if and only if it is continuous.*

Theorem 2.8 ([7])

Let one of the spaces $H^p(\mathbb{U})$ *or* $L^p_a(\mathbb{U})$, *be denoted by* X, $1 \leq p < \infty$ *and* $\alpha > -1(\alpha = -1$ *if* $X = H^p(\mathbb{U}))$, *and let* $\gamma = (\alpha + 2)/p$. *If* $c \in \mathbb{R}$ *and* $\gamma, v \in \mathbb{C}$, *then*

1. *$f(\omega) = (\omega - c)^{\lambda}(\lambda + i)^{v} \in X$ if and only if $Re(\lambda + v) < -\gamma < Re(\lambda)$. In particular, $(\omega - c)^{\lambda} \notin X$ for any $\lambda \in \mathbb{C}$ and $(w + i)^{v} \in X$ if and only if $Re(v) < -\gamma$.*

2. *$f(\omega) = e^{i\omega}/\omega^{c} \in X$ if and only if, $\frac{1}{p} < c < \gamma$. In particular $e^{i\omega}/\omega^{c} \notin H^{p}(\mathbb{U})$ for any $c \in \mathbb{R}$*

Theorem 2.9 (Fatou's lemma)

Let (f_n) be a sequence of non-negative measurable function on a measure space $(X, \mathcal{A}, \mu)$. The Fatou's lemma states

$$\int_{x} lim\, inf f_n d\mu \leq lim\, inf \int_{x} f_n d\mu.$$

Theorem 2.10 (Classification theorem for the $Aut(\mathbb{U})$)

Suppose $\varphi : \mathbb{R} \to Aut(\mathbb{U})$ is a nontrivial continuous group homomorphism. Then only one of the following cases holds:

1. *There exists k in $\mathbb{R}, k \neq 0$ and $g \in Aut(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(g(z) + kt)\ \forall\ z \in \mathbb{U}$ and $t \in \mathbb{R}$ (Translation class).*

2. *There exists $K > 0$, $k \neq 0$, and $g \in Aut(\mathbb{U})$ so that $\varphi_t(z) = g^{-1}(k^t g(z))\ \forall\ z \in \mathbb{U}$ and $t \in \mathbb{R}$ (scaling class).*

3. *There exists $k > 0$, $K \neq 0$, and a conformal mapping g of $\mathbb{U}$ onto $\mathbb{D}$ such that $\varphi_t(z) = g^{-1}(e^{ikt} g(z))\ \forall z \in \mathbb{U}$ and $t \in \mathbb{U}$ and $t \in \mathbb{R}$ (Rotation class).*

Chapter 3

Composition semigroup induced by the scaling class

3.1 Introduction

Following Theorem 2.10, all the automorphism of $(\mathbb{U})$ can be classified into three classes distinctively, these are the translation, the scaling and the rotational classes. As remarked in [7], it is sufficient to consider special cases of the classes:

1. For the scaling class, we consider the automorphism of the following form

$$\varphi_t(z) = e^{-t}z \qquad \forall z \in \mathbb{U},\ t \in \mathbb{R}. \tag{3.1}$$

2. For the translation class, we consider the automorphism of the

following form

$$\varphi_t(z) = z + t \qquad \forall z \in \mathbb{U},\ t \in \mathbb{R}. \tag{3.2}$$

3. For the rotation class, we consider the automorphism of the following form

$$\varphi_t(z) = e^{ikz} \qquad \forall z \in \mathbb{D}, t \in \mathbb{R}. \tag{3.3}$$

Therefore, the corresponding composition semigroup induced by the automorphism classes $(\varphi_t)_{t\geq 0}$ denoted by $\mathcal{C}_{\varphi_t}$ and acting on $\mathcal{D}(\mathbb{U})$ are given by

$$\mathcal{C}_{\varphi_t} g(z) = g(\varphi_t(z)). \tag{3.4}$$

We prove that $\mathcal{C}_{\varphi_t}$ is strongly continuous on $\mathcal{D}(\mathbb{U})$.

Theorem 3.1
Let $\mathcal{C}_{\varphi_t}$ be as defined in (3.4)*. Then the operator $\mathcal{C}_{\varphi_t}$ is strongly continuous on $\mathcal{D}(\mathbb{U})$.*

PROOF. Remember $\|\mathcal{C}_{\varphi_t} g\|^2_{\mathcal{D}\mathbb{U}} = |\mathcal{C}_{\varphi_t} g(i)|^2 + \int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t} g - g)'(z)|^2 dA(z)$. To establish the strong continuity of $(\mathcal{C}_{\varphi_t})_{t\geq 0}$, it is enough to demonstrate that:

$$\lim_{t\to 0^+} \|\mathcal{C}_{\varphi_t} g - g\|_{\mathcal{D}(\mathbb{U})} = 0 \quad \forall g \in \mathcal{D}(\mathbb{U})$$

i.e $|(\mathcal{C}_{\varphi_t} g - g)(i)|^2 + \int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t} g - g)'(z)|^2 dA(z) \to 0$ as $t \to 0^+$. This is equivalent to showing that:

$$|(\mathcal{C}_{\varphi_t} g - g)(i)|^2 \to 0 \text{ as } t \to 0^+ \text{ and}$$

$$\int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t}g - g)(z)|^2 \to 0 \quad \text{as} \quad t \to 0^+.$$

But,

$$|(\mathcal{C}_{\varphi_t}g - g)(i)| \to |(\mathcal{C}_{\varphi_0}g - g)(i)| = 0 \quad \text{as} \quad t \to 0^+$$

To prove that $\lim\limits_{t\to 0^+} \int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t}g - g)'(z)|^2 dA(z) = 0$, we let $g \in \mathcal{D}(\mathbb{U})$ and make a supposition that $t_n \to 0$ in $\mathbb{R}$. We also let $g_n = \mathcal{C}_{\varphi_t}g$. Then $g_n(\omega) \to g(\omega)$ on the compact subsets of $(\mathbb{U})$ and $g_n' = g'$ for each n. Let $f_n(\omega) := 2(|g'|^2) + |g_n'|^2 - |g - g_n'|^2$, then $f_n \geq 0$ and $f_n(\omega) \to 2^2|g'(\omega)|^2$ on $\mathcal{D}(\mathbb{U})$ as $n \to \infty$.
By Fatou's lemma, we have

$$\begin{aligned}
\int_{\mathbb{U}} 2^2|g'(z)|^2 dA &= \int_{\mathbb{U}} \lim_n \inf f_n dA \\
&\leq \lim_n \inf \int_{\mathbb{U}} f_n dA(z) \\
&= \lim_n \inf \int_{\mathbb{U}} 2(|g'|^2 + |g_n'|^2) - (|g' - g_n'|^2) dA(z) \\
&= 2\int_{\mathbb{U}} |g'|^2 dA + 2\int_{\mathbb{U}} |g'|^2 dA - \lim_n \sup \int_{\mathbb{U}} |g' - g_n'| dA(z) \\
&= 2^2 \int_{\mathbb{U}} |g'(z)|^2 - \lim_n \sup \int_{\mathbb{U}} |g' - g_n'| dA(z).
\end{aligned}$$

Thus $0 \leq -\limsup_n \int_{\mathbb{U}} \|g' - g_n'\|^2 dA(z) \leq 0$ implying that $\limsup_n \int_{\mathbb{U}} \|g' - g_n'\|^2 dA(z) = 0$. Hence $\lim_n \int_{\mathbb{U}} \|g' - g_n'\|^2 dA(z) = 0$ that is $\lim_n \int_{\mathbb{U}} \|\mathcal{C}_{\varphi_{tn}} - g\|^2 dA(z) = 0$. Therefore, $\|\mathcal{C}_{\varphi_{tn}} - g\|_{\mathcal{D}(\mathbb{U})} \to 0$ as $t \to 0$ implying that $(\mathcal{C}_{\varphi_t})_{t\in\mathbb{R}}$ is strongly continuous. □

In this chapter, we consider composition semigroup induced by the scaling class given by equation (3.1). Then the composition semigroup induced by the class (3.1) is given by

$$\mathcal{C}_{\varphi_t}g(\omega) = g(e^t w) \quad \forall g \in \mathcal{D}(\mathbb{U}) \tag{3.5}$$

3.2 Semigroup properties

The class $(\mathcal{C}_{\varphi_t})_{t\geq 0}$ given in (3.5) was considered in [15] where it was proved that it forms a group on the Dirichlet space of the upper half plane but it is not an isometry on $\mathcal{D}(\mathbb{U})$. For completion, consider the results below.

Proposition 3.2

Let $\mathcal{C}_{\varphi_t}$ be as given in (3.5). Then $(\mathcal{C}_{\varphi_t})_{t\in\mathbb{R}}$ forms a group on the Dirichlet space of the upper half plane under composition but fails to be an isometry.

For proofs, see [15]

Since $\mathcal{C}_{\varphi_t}$ is not an isometry on $\mathcal{D}(\mathbb{U})$ (use proposition 3.2, assertion 2), Oyugi [15] in her thesis considered the subspace $\mathcal{D}_\circ(\mathbb{U})$ of $\mathcal{D}(\mathbb{U})$ which consists of functions that vanish at the point i, that is, $g(i) = 0$, and therefore $\mathcal{D}_\circ(\mathbb{U}) = \{g \in \mathcal{D}(\mathbb{U}) : g(i) = 0\}$. The norm on $\mathcal{D}_{\circ(\mathbb{U})}$ is therefore given by

$$\|g\|^2_{\mathcal{D}(\mathbb{U})} = \int_{\mathbb{U}} |g'(\omega)|^2 dA(\omega).$$

However, in this setting, the operator $\mathcal{C}_{\varphi_t}$ does not map $\mathcal{D}_\circ(\mathbb{U})$ on itself as required for these semigroups. In this regard, the operator $\mathcal{C}_{\varphi_t}$ is redefined as

$$\hat{\mathcal{C}}_{\varphi_t} g(z) = g(e^{-t}z) - g(e^{-t}i) \quad \forall g \in \mathcal{D}_\circ(\mathbb{U}) \tag{3.6}$$

so that $\hat{\mathcal{C}}_{\varphi_t} g(i) = g(e^{-t}i) - g(e^{-t}i) = 0$. Now $\hat{\mathcal{C}}_{\varphi_t}$ maps $\mathcal{D}_\circ(\mathbb{U})$ into itself, that is, $\hat{\mathcal{C}}_{\varphi_t} : \mathcal{D}_\circ(\mathbb{U}) \to \mathcal{D}_\circ(\mathbb{U})$

The property of the class operator $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ was partially investigated in [15] and we summarize them in the next proposition. However, for completeness of this work, we include our own proofs.

Proposition 3.3

Let $\hat{\mathcal{C}}_{\varphi_t}$ be as defined in (3.6). *Then:*

1. *The class $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ defines a group on $\mathcal{D}_\circ(\mathbb{U})$.*

2. *$\hat{\mathcal{C}}_{\varphi_t}$ is a linear isometry on $\mathcal{D}_\circ(\mathbb{U})$.*

3. *The operator $\hat{\mathcal{C}}_{\varphi_t}$ is strongly continuous on $\mathcal{D}_\circ(\mathbb{U})$.*

4. *The infinitesimal generator Γ of $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ is given by*

$$\Gamma g(z) = -zg'(z) + ig'(i) \tag{3.7}$$

with dom$(\Gamma) = \{g \in \mathcal{D}_\circ(\mathbb{U}) : zg'(z) \in \mathcal{D}(\mathbb{U})\}$.

PROOF. To proof (1), it corrects both $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ and $(\hat{\mathcal{C}}_{\varphi_t})_{t\leq 0}$ as semigroups on $\mathcal{D}_\circ(\mathbb{U})$.

Now for all $g \in \mathcal{D}_\circ(\mathbb{U})$

$$\begin{aligned} \hat{\mathcal{C}}_{\varphi_0} &= g(e^0\omega) - g(e^0 i) \\ &= g(\omega) - g(i) \\ &= g(\omega). \end{aligned}$$

This implies that $\hat{C}_{\varphi_0} = I$, the identity operator on $\mathcal{D}_\circ(\mathbb{U})$. Next, we have for all $t, s \geq 0$;

$$\begin{aligned}
(\hat{C}_{\varphi_t} \circ \hat{C}_{\varphi_s})g(\omega) &= \hat{C}_{\varphi_t}(\hat{C}_{\varphi_s} g(\omega)) \\
&= \hat{C}_{\varphi_t}(g(e^{-s}\omega) - g(e^{-s}i)) \\
&= g(e^{-t}(e^{-s}\omega)) - g(e^{-t}(e^{-s}i)) \\
&= g(e^{-(s+t)}\omega) - g(e^{-(s+t)}i) \\
&= \hat{C}_{\varphi_{s+t}} g(\omega),
\end{aligned}$$

as desired.

Hence, $(\hat{C}_{\varphi_t})_{t\geq 0}$ forms a semigroup on $\mathcal{D}_\circ(\mathbb{U})$. Applying the same argument, proves that $(\hat{C}_{\varphi_t})_{t\leq 0}$ also forms a semigroup on $\mathcal{D}_\circ(\mathbb{U})$. For this reason the class $(\hat{C}_{\varphi_t})_{t\in\mathbb{R}}$ forms a group on $\mathcal{D}_\circ(\mathbb{U})$. This proves assertion 1.

For assertion 2, we first show that $\hat{C}_{\varphi_t}$ is linear. $\forall f, g \in \mathcal{D}_\circ(\mathbb{U})$ and $\alpha, \beta \in \mathbb{F}$, we have;

$$\begin{aligned}
\hat{C}_{\varphi_t}(\alpha f + \beta g)(\omega) &= (\alpha f + \beta g)(e^{-t}\omega) - (\alpha f + \beta g)(e^{-t}i) \\
&= \alpha f(e^{-t}\omega) + \beta g(e^{-t}\omega) - \alpha f(e^{-t}i) - \beta g(e^{-t}i) \\
&= \alpha(f(e^{-t}\omega) - f(e^{-t}i)) + \beta(g(e^{-t}\omega) - g(e^{-t}i)) \\
&= \alpha\hat{C}_{\varphi_t} f(\omega) + \beta\hat{C}_{\varphi_t} g(\omega)
\end{aligned}$$

as desired.

Next, we prove that $\hat{\mathcal{C}}_{\varphi_t}$ is an isometry. Now, $\forall g \in \mathcal{D}_\circ(\mathbb{U})$ we have;

$$\begin{aligned}
\|\hat{\mathcal{C}}_{\varphi_t} g\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |(\hat{\mathcal{C}}_{\varphi_t}(g))'(z)|^2 dA(z) \\
&= \int_{\mathbb{U}} |(g(e^{-t}z) - g(e^{-t}i))'|^2 dA(z) \\
&= \int_{\mathbb{U}} |e^{-t} g'(e^{-t}z)|^2 dA(z).
\end{aligned}$$

Changing variables, let $\omega = e^{-t}z$ then $z = e^t\omega$ and applying the Jacobian, it implies that $dA(z) = e^{2t}dA(\omega)$. Therefore:

$$\begin{aligned}
\|\hat{\mathcal{C}}_{\varphi_t} g\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |e^{-t} g'(e^{-t}z)|^2 dA(z) \\
&= \int_{\mathbb{U}} e^{-2t} |g'(\omega)|^2 e^{2t} dA(\omega) \\
&= \int_{\mathbb{U}} |g'(\omega)|^2 dA(\omega) \\
&= \|g\|^2_{\mathcal{D}_\circ(\mathbb{U})}.
\end{aligned}$$

The proof of strong continuity follows from Theorem 3.1.

Next we find Γ of the group $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$.

By definition, the infinitesimal generator Γ of the group $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ is given by;

$$\begin{aligned}
\Gamma g(z) &= \lim_{t\to 0^+} \frac{(g(e^t z) - g(e^{-t}i)) - g(z)}{t} \\
&= \left.\frac{\partial}{\partial t}(g(e^{-t}z) - g(e^{-t}i))\right|_{t=0} \\
&= (-e^{-t}zg'(e^{-t}z) + ie^{-t}g'(e^{-t}i))\big|_{t=0} \\
&= -zg'(z) + ig'(i).
\end{aligned}$$

Therefore $\Gamma g(z) = -zg'(z) + ig'(i)$. This implies that $\mathrm{dom}(\Gamma) \subseteq \{g \in \mathcal{D}_\circ(\mathbb{U}) : zg'(z) \in \mathcal{D}(\mathbb{U})\}$. Conversely, we let $g \in \mathcal{D}_\circ(\mathbb{U})$ such that

$zg'(z) \in \mathcal{D}(\mathbb{U})$. Then $\forall z \in \mathbb{U}$, we have,

$$\begin{aligned}
\hat{\mathcal{C}}_{\varphi_t}g(z) - g(z) &= \int_0^t \frac{\partial}{\partial s}(g(e^{-s}z) - g(e^{-s}i))ds \\
&= \int_0^t -e^{-s}zg'(e^{-s}z) + ie^{-s}g'(e^{-s}i)ds \\
&= \int_0^t e^{-s}(-zg'(z) + ig(i))ds \\
&= \int_0^t \hat{\mathcal{C}}_{\varphi_s}G(z)ds,
\end{aligned}$$

where $G(z) = -zg'(z) + ig'(i)$. Thus $\lim\limits_{t\to 0}\frac{\hat{\mathcal{C}}_{\varphi_t}g-g}{t} = \lim\limits_{t\to 0}\frac{1}{t}\int_0^t \mathcal{C}_{\varphi_s}(G)ds$ and strong continuity of $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ implies that $\frac{1}{t}\int_0^t \|\hat{\mathcal{C}}_{\varphi_t}G - G\|ds \to 0$ as $t \to 0$. Hence $\mathrm{dom}(\Gamma) \supseteq \{g \in \mathcal{D}(\mathbb{U}) : zg'(z) \in \mathcal{D}(\mathbb{U})\}$, which completes our proof. $\square$

3.3 Spectral properties

In this section we determine the spectrum of the semigroup $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$. We begin by proving the following Lemma that highlights different scenarios for an analytic function g.

Lemma 3.4

1. *If $c \in \mathbb{R}$ and $\lambda, v \in \mathbb{C}$, then $g(\omega) = (\omega - c)^{\lambda}(\omega + i)^{v} \in L_a^2(\mathbb{U})$ if and only if $Re(\lambda + v) < -1 < Re(\lambda)$. In particular, $(\omega - c)^{\lambda} \notin L_a^2(\mathbb{U})$ for any $\lambda \in \mathbb{C}$ and $(\omega + i)^{v}$ if and only if $Re(v) < -1$.*

2. *$g(\omega) = \frac{e^{i\omega}}{\omega^c} \in L_a^2(\mathbb{U})$ if and only if $\frac{1}{2} < c < 1$.*

3. *$g \in \mathcal{D}_\circ(\mathbb{U})$ if and only if $g' \in L_a^2(\mathbb{U})$.*

4. *$g' \in L_a^2(\mathbb{U})$ if and only if $\psi' g' \circ \psi \in L_a^2(\mathbb{D})$.*

5. $g \in \mathcal{D}_\circ(\mathbb{U})$ *if and only if* $\psi' g' \circ \psi \in L^2_a(\mathbb{D})$.

PROOF. Assertion 1 and 2 results from Theorem 2.8 by taking $p = 2$ and $\alpha = 0$. Assertion 3 follows from the definition. Indeed, $g \in \mathcal{D}_\circ(\mathbb{U})$ if and only if $\int_{\mathbb{U}} |g'(z)|^2 dA(z) < \infty$ which is equivalent to $g' \in L^2_a(\mathbb{U})$. This proves assertion 3.
Now, let $S_\psi g := \psi' g \circ \psi$, where ψ is the Cayley transform. By a change of variables argument: let $\omega = \psi(z)$, then $dA(\omega) = |\psi'(z)|^2 dA(z)$ where $|\psi'(z)|^2$ is the Jacobian.
Therefore,

$$\begin{aligned}
\|S_\psi g\|^2_{L^2_a(\mathbb{D})} &= \int_{\mathbb{U}} |\psi'(z)|^2 |g(\psi(z))|^2 dA(z) \\
&= \int_{\mathbb{U}} |g(\omega)|^2 dA(\omega) \\
&= \|g\|^2_{L^2_a(\mathbb{U})},
\end{aligned}$$

which proves assertion 4.
Assertion 5 follows clearly from assertions 3 and 4. Indeed $g \in \mathcal{D}_\circ(\mathbb{U}) \Leftrightarrow g' \in L^2_a(\mathbb{U}) \Leftrightarrow \psi' g' \circ \psi \in L^2_a(\mathbb{D})$. $\square$
Next, we fully determine the spectral properties of the infinitesimal generator of $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ on $\mathcal{D}_\circ(\mathbb{U})$.

Theorem 3.5
Suppose Γ *is the infinitesimal generator of the group* $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ *on* $\mathcal{D}_\circ(\mathbb{U})$ *given in equation* (3.7)*, then the point spectrum* $\sigma_p(\Gamma)$ *of* Γ *is empty, that is* $\sigma_p(\Gamma) = \emptyset$ *and the spectrum* $\sigma(\Gamma)$ *of* Γ *is exactly in the imaginary line, that is,* $\sigma(\Gamma) = i\mathbb{R}$. *In particular,* Γ *is unbounded.*

PROOF. To obtain the point spectrum of Γ, we let λ be an eigenvalue of Γ and let g be a corresponding eigenvector, such that $\Gamma g = \lambda g$. This implies that $-zg'(z) + ig'(i) = \lambda g(z)$, which can be expresses as

$$-zg'(z) - \lambda g(z) = -D$$

where $D = ig'(i)$. Using integrating factor technique, we obtain a general equation,

$$g(z) = \frac{D}{\lambda} + Cz^{-\lambda},$$

where $D = ig'(i)$ and C is an arbitrary constant. Through differentiation, we find $g'(z) = -\lambda C^{-(\lambda+1)}$. According to Lemma 3.4 that $g' \in L^2_a(\mathbb{U})$ if and only if $\Re(\lambda) < -1 < \Re(\lambda)$. However, there is no such λ that satisfies this condition, leading to $\sigma_p(\Gamma) = \emptyset$.
Next we show that $\sigma(\Gamma) = i\mathbb{R}$.
Since each $(\hat{\mathcal{C}}_{\varphi_t})$ is an invertible isometry, (Theorem 2.3) its spectrum satisfies $\sigma(\hat{\mathcal{C}}_{\varphi_t}) \subseteq \partial\mathbb{D}$, and the spectral mapping theorem for strongly continuous group, (Theorem 2.6) implies that $e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{\mathcal{C}}_{\varphi_t})$. Therefore,

$$e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{\mathcal{C}}_{\varphi_t}) \subseteq \partial\mathbb{D}.$$

Let $\lambda \in \sigma(\Gamma)$, then $|e^{\lambda t}| = 1$ since $e^{t\sigma(\Gamma)} \subseteq \partial\mathbb{D}$.
This means that

$$\begin{aligned} e^{t\Re(\lambda)} = 1 \quad &\Rightarrow \quad t\Re(\lambda) = 0 \\ &\Rightarrow \quad \Re(\lambda) = 0. \end{aligned}$$

This shows that $\lambda \in i\mathbb{R}$ which implies that $\sigma(\Gamma) \subseteq i\mathbb{R}$.
To prove that $i\mathbb{R} \supseteq \sigma(\Gamma)$, fix $\lambda \in i\mathbb{R}$ and consider the function $h(\omega) :=$

$h_1(\omega) - h_1(i)$ where $h_1(\omega) = \omega^{-\lambda+1}(\omega+i)^{-2}$. By assertion 1 and Lemma 3.4 $h \in \mathcal{D}_\circ(\mathbb{U})$ since $h'(\omega) = \omega^{-\lambda}(\omega+i)^{-2} + \omega^{-\lambda+1}(\omega+i)^{-3}$ and $Re(-\lambda-2) = -2$, $Re(-\lambda+1-3) = -2$ which is less than -1 as required in assertion 1 and lemma 3.4.

We consider equation

$$(\lambda - \Gamma)g = h.$$

Substituting the values of Γ and h we get,

$$\lambda g(\omega) + \omega g'(\omega) - ig'(i) = \omega^{-\lambda+1}(\omega+i)^{-2} + A, \tag{3.8}$$

where $A = h_1(i)$.

To solve (3.8), let $A + ig'(i)$ be B so that,

$$\lambda g(\omega) + \omega g'(\omega) = \omega^{-\lambda+1}(\omega+i)^{-2} + B.$$

Dividing through by ω yields

$$g'(\omega) + \frac{\lambda}{\omega}g(\omega) = \frac{\omega^{-\lambda+1}(\omega+i)^{-2}}{\omega} + \frac{B}{\omega}.$$

This forms the first order differential equation which can be solved by applying the integrating factor method.

$$g(\omega) = -\omega^{-\lambda}(\omega+i)^{-1} + \frac{B}{\lambda} + C\omega^{-\lambda}, \tag{3.9}$$

where $B = ig(i)$ and C is an arbitrary constant.

Checking if g given by equation (3.9) is in $\mathcal{D}(\mathbb{U})$, we find g'. By differentiation,

$$g'(\omega) = \omega^{-\lambda}(\omega+i)^{-2} + \lambda\omega^{-(\lambda+1)}(\omega+i)^{-1} - \lambda C\omega^{-(\lambda+1)}. \quad (3.10)$$

Using Lemma 3.4, assertion 3 and 4, it suffices to check if $\psi'(z)g'(\psi(z))$ is in $L_a^2(\mathbb{D})$.

Since

$$\psi(z) = \frac{i(1+z)}{1-z},$$

it follows that,

$$\psi'(z) = \frac{2i}{(1-z)^2}. \quad (3.11)$$

Also, using (3.10)

$$\psi'(z)g'(\psi(z)) = \psi'(z)\left[(\psi(z))^{-\lambda}(\psi(z)+i)^{-2} + \lambda\psi(z)^{-(\lambda+1)}(\psi(z)+i)^{-1} - \lambda C(\psi(z))^{-(\lambda+1)}\right]. \quad (3.12)$$

Now, using (3.11) and (3.12) we obtain

$$\psi'(z)g'(\psi'(z)) = \frac{2i}{(1-z)^2}\left[\frac{(1-z)^{\lambda}}{i^{\lambda}(1+z)^{\lambda}}(\frac{i(1+z)}{1-z}+i)^{-2} + \lambda\frac{(1-z)^{\lambda+1}}{i^{\lambda+1}(1+z)^{\lambda+1}}(\frac{i(1+z)}{1-z}+i)^{-1} - \lambda C\frac{(1-z)^{\lambda+1}}{i^{\lambda+1}(1+z)^{\lambda+1}}\right].$$

Simplifying, we have

$$\psi'(z)g'(\psi(z)) = \frac{(1-z)^{\lambda}}{i^{\lambda}(1+z)^{\lambda}}\left[\frac{-i}{2} + \frac{\lambda}{i(1+z)} - \frac{2\lambda C}{1-z^2}\right].$$

Thus;

$$\psi'(z)g'(\psi(z)) = C_1(1-z)^{\lambda}(1+z)^{-\lambda} + C_2(1-z)^{\lambda-1}(1+z)^{-(\lambda+1)} + CC_3(1-z)^{\lambda-1}(1+z)^{-(\lambda+1)}.$$

Take $f(z) := \psi'(z)g'(\psi(z))$. Then $f \in L^2_a(\mathbb{D})$ if and only if all these conditions are satisfied: $Re(\lambda) > -1, Re(-\lambda) > 0, Re(\lambda) > 0$, and $Re(-\lambda) > -1$, which is impossible and therefore $f \notin L^2_a(\mathbb{U})$, implying that $g \notin \mathcal{D}(\mathbb{U})$. Thus $h \notin ran(\lambda - \Gamma)$, that is $\lambda - \Gamma$ is not invertible and so $\lambda \in \sigma(\Gamma)$. Hence $i\mathbb{R} \subseteq \sigma(\Gamma)$, as desired. □

REMARK 3.6

In Theorem 3.5 above, we have fully computed the spectrum of the semigroup $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$. This is an extension of the work by [15] where the spectrum was partially determined. As a result,we obtain the entire class of integral operators as resolvents of Γ of this group. In [15], only a single integral operator was obtained . This study therefore completes the spectral analysis of the composition semigroup induced by scaling class.

3.4 A class of integral operators and its properties

In this section, we obtain the resolvents as a class of integral operators on $\mathcal{D}_\circ(\mathbb{U})$. Further, we determine the spectra, the norm and the spectral radius of the resolvent operator obtained.

Theorem 3.7

If Γ is the infinitesimal generator of $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ on the $\mathcal{D}_\circ(\mathbb{U})$, then the following holds;

1. *If $Re(v) > 0$ then*

$$R(v,\lambda)h(z) = z^{-v}\int_0^z \omega^{v-1}\hat{h}(\omega)d\omega,$$

   where $\hat{h}(\omega) = h(\omega) - h(\frac{\omega}{z}i)$.

2. $\sigma(R(v,\Gamma)) = \left\{\omega : \left|\omega - \frac{1}{2\Re(v)}\right| = \frac{1}{|2\Re(v)|}\right\}.$

3. $r(R(v,\Gamma)) = \frac{1}{|\Re(v)|}.$

4. $\|R(v,\Gamma)\| = \frac{1}{|\Re(v)|}.$

PROOF. Following from Theorem 3.5, $\sigma_p(\Gamma) = \emptyset$ and $\sigma(\Gamma) = i\mathbb{R}$. Now, for $v \in \mathbb{C}$, if $Re(v) > 0$, then $v \in \rho(\Gamma)$ and the resolvent operator $R(v,\Gamma)$, is given in Theorem 2.2: $\forall h \in \mathcal{D}_\circ(\mathbb{U})$,

$$\begin{aligned} R(v,\Gamma)h(z) &= \int_0^\infty e^{-vt}\hat{\mathcal{C}}_{\varphi_t}h(z)dt \\ &= \int_0^\infty e^{-vt}[h(e^{-t}z) - h(e^{-t}i)]dt. \end{aligned}$$

Changing the variable and letting $\omega = e^{-t}z$, then $e^{-t} = \frac{\omega}{z}$, and so $e^t = \frac{z}{\omega}$, $d\omega = -e^{-t}zdt$ and $dt = -\frac{1}{\omega}d\omega$; $t = 0 \Rightarrow \omega = z$, $t = \infty \Rightarrow \omega = 0$ and also $t = ln\, z - ln\, \omega$.

Therefore:

$$\begin{aligned} R(v,\Gamma)h(z) &= \int_0^z e^{-v(ln\,z - ln\,\omega)}[h(\omega) - h(\frac{\omega}{z}i)](-\frac{1}{\omega}d\omega) \\ &= \int_z^0 z^{-v}\omega^v[h(\omega) - h(\frac{\omega}{z}i)]\frac{1}{\omega}d\omega \\ &= z^{-v}\int_0^z \omega^{v-1}\hat{h}(\omega)d\omega. \end{aligned}$$

If $Re(v) < 0$, then $-v \in \rho(-\Gamma)$, where $-\Gamma$ is the generator of $\hat{\mathcal{C}}_{\varphi_{-t}}h(z) = h(e^t z) - h(e^t i)$, $t \geq 0$.

Therefore:

$$\begin{aligned} R(v,\Gamma)h &= -R(-v,-\Gamma)h \\ &= \int_0^\infty e^{vt}\hat{\mathcal{C}}_{\varphi_{-t}}h dt. \end{aligned}$$

Changing the variables, let $\omega = e^t z$, then $e^t = \frac{\omega}{z}$; $\frac{d\omega}{dt} = \omega \Rightarrow dt = \frac{1}{\omega}$; $t = ln\,\omega - ln\,z$; $t = 0 \Rightarrow \omega = z; t = \infty \Rightarrow \omega = \infty$.

We have:

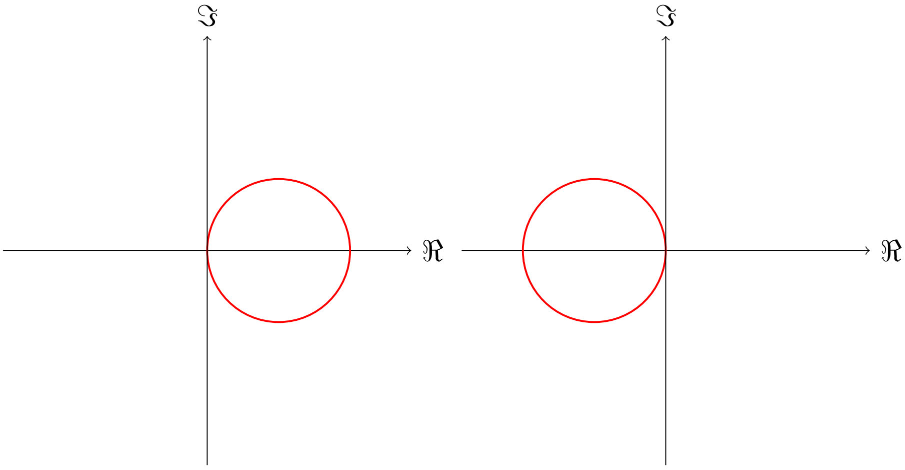
$$\begin{aligned} R(v,\Gamma)h(z) &= -R(-v,-\Gamma)h(z) \\ &= -\int_z^\infty e^{v(ln\,\omega - ln\,z)}[h(\omega) - h(\frac{\omega}{z}i)]\frac{1}{\omega}d\omega \\ &= -z^{-v}\int_z^\omega \omega^{v-1}\hat{h}(\omega)d\omega, \end{aligned}$$

where $\hat{h}(\omega) = h(\omega) - h(\frac{\omega}{z}i)$. This proves claim 1.

We let $\mathcal{C}_v := R(v, \Gamma)$. Then Theorem 2.5 implies that

$$\begin{aligned}\sigma(\mathcal{C}_v) &= \left\{\frac{1}{v-\gamma} : \gamma \in \sigma(\Gamma)\right\} \cup \{0\} \\ &= \left\{\frac{1}{v-is} : s \in \mathbb{R}\right\} \cup \{0\} \\ &= \left\{\omega \in \mathbb{C} : \left|\omega - \frac{1}{2\Re(v)}\right| = \frac{1}{|2\Re(v)|}\right\}.\end{aligned}$$

Graphical representation of the spectrum $\sigma(\mathcal{C}_v)$ is as given below:



(a) $\Re(v) > 0$ (b) $\Re(v) < 0$

For 3, by definition we have,

$$\begin{aligned}r(R(v, \Gamma)) &= \sup\{|\omega| : \omega \in \sigma(R(v, \lambda))\} \\ &= \sup\left\{|\omega| : \left|\omega - \frac{1}{2\Re(v)}\right| = \frac{1}{|2\Re(v)|}\right\}\end{aligned}$$

which implies that;

$$\begin{aligned}|\omega| &\leq \frac{1}{|2\Re(v)|} + \frac{1}{|2\Re(v)|} \\ &= \frac{1}{|\Re(v)|}.\end{aligned}$$

Therefore, $r(R(v,\Gamma)) = \sup\left\{|\omega| : |\omega| \leq \frac{1}{|\Re(v)|}\right\}=\frac{1}{|\Re(v)|}$.

Finally, we prove 4.

Since a norm bounds an operator's spectral radius then,

$$r(R(v,\Gamma)) \leq \|R(v,\Gamma)\|$$

which implies that

$$\frac{1}{|Re(v)|} = r(R(v,\Gamma)) \leq \|R(v,\Gamma)\|.$$

Theorem 2.4 asserts that

$$\|R(v,\Gamma)\| \leq \frac{1}{|\Re(v)|}.$$

Thus,

$$\frac{1}{|\Re(v)|} = r(R(v,\Gamma)) \leq \|R(v,\Gamma)\| \leq \frac{1}{|\Re(v)|}.$$

So, $|R(v,\Gamma)\| = \frac{1}{|\Re(v)|}$, as desired. □

3.5 Adjoint of the integral operator

In this section, we have determined the adjoint of the composition semigroup on $\mathcal{D}_\circ(\mathbb{U})$.

Theorem 3.8

Let $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ *be given by* (3.6) *on* $\mathcal{D}_\circ(\mathbb{U})$ *and* Γ *be its infinitesimal generator given by* (3.7)*. Then the following holds:*

1. $\hat{\mathcal{C}}^*_{\varphi_t} = \hat{\mathcal{C}}_{\varphi_{-t}}$.

2. $(\hat{\mathcal{C}}^*_{\varphi_t})_{t\geq 0}$ *is also a group of linear isometries on* $\mathcal{D}_\circ(\mathbb{U})$.

3. $\hat{\mathcal{C}}^*_{\varphi_t}$ *is strongly continuous on* $\mathcal{D}_\circ(\mathbb{U})$.

4. *The infinitesimal generator* Γ^* *of* $(\hat{\mathcal{C}}^*_{\varphi_t})_{t\geq 0}$ *is given by*

$$\Gamma^* f(\omega) = \omega f'(\omega) - i f'(i) \tag{3.13}$$

with $dom(\Gamma^*) = \{f \in \mathcal{D}_\circ(\mathbb{U}) : \omega f'(\omega) \in \mathcal{D}(\mathbb{U})\}$. *In particular,* $\Gamma^* = -\Gamma$ *and* $dom(\Gamma) = dom(\Gamma^*)$.

PROOF. Since $\mathcal{D}_\circ(\mathbb{U})$ is a Hilbert space, we have that $\forall g, f \in \mathcal{D}_\circ(\mathbb{U})$

$$\begin{aligned}\left\langle \hat{\mathcal{C}}_{\varphi_t} g, f \right\rangle &= \int_{\mathbb{U}} (\hat{\mathcal{C}}_{\varphi_t} g)'(t)\overline{f'(z)}dA(z)\\ &= \int_{\mathbb{U}} e^{-t} g'(e^{-t}z)\overline{f'(z)}dA(z).\end{aligned}$$

Changing the variables, let $\omega = e^{-t}z$. Then $dA(\omega) = e^{-2t}dA(z)$ and therefore,

$$\begin{aligned}\left\langle \hat{\mathcal{C}}_{\varphi_t} g, f \right\rangle &= \int_{\mathbb{U}} e^{-t} g'(\omega)\overline{f'(e^t\omega)}e^{2t}dA(\omega)\\ &= \int_{\mathbb{U}} g'(\omega)\overline{f'(e^t\omega)}e^{t}dA(\omega)\\ &= \int_{\mathbb{U}} g'(\omega)\overline{e^t f'(e^t\omega)}dA(\omega)\\ &= \int_{\mathbb{U}} g'(\omega)\overline{(f(e^t\omega) - f(e^t i))}dA(\omega)\\ &= \int_{\mathbb{U}} g'(\omega)(\hat{\mathcal{C}}_{\varphi_{t-t}} f)'(\omega)dA(\omega)\\ &= \left\langle g, \hat{\mathcal{C}}^*_{\varphi_t} f \right\rangle,\end{aligned}$$

as desired. Therefore $\hat{\mathcal{C}}^*_{\varphi_t} = \hat{\mathcal{C}}_{\varphi_{-t}}$ where $\hat{\mathcal{C}}^*_{\varphi_t} f = f(e^t\omega) - f(e^t i)$. This completes the prove for assertion 1.

For assertion 2, we first prove that $\hat{C}^*_{\varphi_t}$ is linear. $\forall f, g \in \mathcal{D}_\circ(\mathbb{U})$ and $\alpha, \beta \in \mathbb{F}$ we have;

$$\begin{aligned}\hat{C}^*_{\varphi_t}(\alpha f + \beta g)(z) &= (\alpha f + \beta g)(e^t z) - (\alpha f + \beta g)(e^t i)\\ &= \alpha f(e^t z) + \beta g(e^t z) - \alpha f(e^t i) - \beta g(e^t i)\\ &= \alpha(f(e^t z) - f(e^t i)) + \beta(g(e^t z) - g(e^t i))\\ &= \alpha \hat{C}^*_{\varphi_t} f(z) + \beta \hat{C}_{\varphi_t} g(z),\end{aligned}$$

as desired.

Next we prove that $\hat{C}^*_{\varphi_t}$ is an isometry. Now, $\forall f \in \mathcal{D}_\circ(\mathbb{U})$, we have

$$\begin{aligned}\|\hat{C}^*_{\varphi_t} f\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |(\hat{C}^*_{\varphi_t} f)'(z)|^2 dA(z)\\ &= \int_{\mathbb{U}} |(f(e^t z) - f(e^t i))'|^2 dA(z)\\ &= \int_{\mathbb{U}} |e^t f'(e^t z)|^2 dA(\omega)\end{aligned}$$

Changing the variables, let $\omega = e^t z$ then $z = e^t \omega$, applying the Jacobian, it implies that $dA(z) = e^{-2t} dA(\omega)$. Therefore:

$$\begin{aligned}\|\hat{C}^*_{\varphi_t} f\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |e^t f'(e^t z)|^2 dA(z)\\ &= \int_{\mathbb{U}} e^{2t} |f'(\omega)|^2 e^{-2t} dA(\omega)\\ &= \int_{\mathbb{U}} |f'(\omega)|^2 dA(\omega)\\ &= \|f\|^2_{\mathcal{D}_\circ(\mathbb{U})}.\end{aligned}$$

Strong continuity of $\hat{C}^*_{\varphi_t}$ follows from strong continuity of $\hat{C}_{\varphi_t}$.

Next we prove assertion 4.

By definition, Γ^* of the group $(\hat{\mathcal{C}}^*_{\varphi_t})_{t\geq 0}$ is;

$$\begin{aligned}\Gamma^* f(\omega) &= \lim_{t\to 0^+}\frac{(f(e^t\omega)-f(e^t i))-f(\omega)}{t}\\ &= \frac{\partial}{\partial t}(f(e^t\omega)-f(e^t i))\Big|_{t=0}\\ &= e^t\omega f'(e^t\omega)-ie^t f'(e^t i)|_{t=0}\\ &= \omega f'(\omega)-if'(i).\end{aligned}$$

$\therefore\ \Gamma^* f'(\omega) = \omega f'(\omega)-if'(i)$. This implies that $\mathrm{dom}(\Gamma)\subseteq\{f\in\mathcal{D}_\circ(\mathbb{U}):\omega f'(\omega)\in\mathcal{D}(\mathbb{U})\}$.

Conversely, we let $f\in\mathcal{D}_\circ(\mathbb{U})$ such that $\omega f'(\omega)\in\mathcal{D}(\mathbb{U})\,\forall z\in\mathbb{U}$ we have

$$\begin{aligned}\hat{\mathcal{C}}_{\varphi_t} f(\omega)-f(\omega) &= \int_0^t \frac{\partial}{\partial s}(f(e^s\omega)-f(e^s i))ds\\ &= \int_0^t e^s\omega f'(e^s\omega)-ie^s f'(e^s i)ds\\ &= \int_0^t e^s(\omega f'(\omega)-if'(i))ds\\ &= \int_0^t \hat{\mathcal{C}}_{\varphi_s}F(\omega)ds,\end{aligned}$$

where $F(\omega)=\omega f'(\omega)-if'(i)$. Thus $\lim_{t\to 0}\frac{\hat{\mathcal{C}}_{\varphi_t}f-f}{t} = \lim_{t\to 0}\frac{1}{t}\int_0^t \mathcal{C}_{\varphi_s}(F)ds$ and strong continuity of $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ implies that $\frac{1}{t}\int_1^t\int_0^t\|\hat{\mathcal{C}}_{\varphi_t}F-F\|ds\to 0$ as $t\to 0$. Hence $\mathrm{dom}(\Gamma)\subseteq\{f\in\mathcal{D}_\circ(\mathbb{U}):\omega f'(\omega)\in\mathcal{D}(\mathbb{U})\}$, which gives the result. □

Theorem 3.9

Let $\hat{\mathcal{C}}^*_v$ be the adjoint of $\mathcal{C}_v$. Then $\mathcal{C}^*_v=-\mathcal{C}_{-v}$.

PROOF. For $v\in\rho(\Gamma)$, we have $R(v,\Gamma)^*=R(v,\Gamma^*)=R(v,-\Gamma)$. Therefore:

For $Re(v) > 0$, we have

$$\begin{aligned} \mathcal{C}_v^* &= R(v, \Gamma)^* \\ &= R(v, -\Gamma) \\ &= -R(-v, \Gamma) \\ &= -\mathcal{C}_{-v} \end{aligned}$$

For $Re(v) < 0$ then,

$$\begin{aligned} \mathcal{C}_v^* &= R(v, \Gamma)^* \\ &= -R(-v, -\Gamma)^* \\ &= -R(-v, -\Gamma^*) \\ &= -R(-v, \Gamma) \\ &= -\mathcal{C}_{-v}. \end{aligned}$$

This completes the proof. $\square$

Chapter 4

Composition semigroup induced by the translation class

In this chapter we consider the composition semigroups induced by the translation class given by equation (3.2). Then the composition semigroup induced by the class (3.2) on $\mathcal{D}_\circ(\mathbb{U})$ is given by

$$\mathcal{C}_{\varphi_t} g(z) = g(z+t), \tag{4.1}$$

for all $g \in \mathcal{D}(\mathbb{U})$.

We also determined the spectral and semigroup properties of equation (4.1). Finally, we constructed an integral operator on $\mathcal{D}_\circ(\mathbb{U})$ that we attain as the resolvent of infinitesimal generator at a point $\lambda = 1$ then partially determine the norm and the spectral properties of the operator.

4.1 Semigroup properties

This section dwels on the study of semigroup properties of composition semigroup (4.1). In particular, we prove that it is a group, but fails to be an isometry and it is strongly continuous on $\mathcal{D}(\mathbb{U})$. Moreover we determine Γ.

Proposition 4.1

Let $\mathcal{C}_{\varphi_t}$ *be as given in equation* (4.1). *Then the functions* $(\mathcal{C}_{\varphi_t})_{t\in\mathbb{R}}$ *forms a group on Dirichlet space of the upper half plane under composition.*

PROOF.

$$\begin{aligned} \mathcal{C}_{\varphi_0} g(z) &= g(z+0) \\ &= g(z), \end{aligned}$$

which shows that $\mathcal{C}_{\varphi_0} = I$

Next, we have for all $t, s \geq 0$;

$$\begin{aligned} \mathcal{C}_{\varphi_t} \circ \mathcal{C}_{\varphi_s} g(z) &= \mathcal{C}_{\varphi_t}(g(z+s)) \\ &= g(z+t+s) \\ &= g(z+(t+s)) \\ &= \mathcal{C}_{\varphi_{t+s}} g(z). \end{aligned}$$

Therefore $\mathcal{C}_{\varphi_{t+s}} g(z) = \mathcal{C}_{\varphi_t} \circ \mathcal{C}_{\varphi_s} g(z)$. Thus $(\mathcal{C}_{\varphi_t})_{t\geq 0}$ is a semigroup on $\mathcal{D}(\mathbb{U})$.

In a similar way it is clear that $(\mathcal{C}_{\varphi t})_{t\leq 0}$ forms a semigroup on Dirichlet space of the upper half plane. Therefore $(\mathcal{C}_{\varphi_t})_{t\in\mathbb{R}}$ is a group under composition as intended. □

In the following proposition we assert that the operator $(\mathcal{C}_{\varphi_t})_{t\in\mathbb{R}}$ fails to be an isometry on Dirichlet space of the upper half plane.

Proposition 4.2

The operator $\mathcal{C}_{\varphi_t}$, given by (4.1) *fails to be an isometry on $\mathcal{D}(\mathbb{U})$.*

PROOF. From the definition of the norm,

$$\|\mathcal{C}_{\varphi_t}g\|^2_{\mathcal{D}(\mathbb{U})} = |\mathcal{C}_{\varphi_t}g(i)|^2 + \int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t}g)'(z)|^2 dA(z).$$

But $\mathcal{C}_{\varphi_t}g(z) = g(z+t)$. Therefore $|\mathcal{C}_{\varphi_t}g(i)|^2$ becomes $|\mathcal{C}_{\varphi_t}g(i+t)|^2$. Implying that $|\mathcal{C}_{\varphi_t}g(i)|^2 = |g'(z+t)|^2$. Therefore, $(\mathcal{C}_{\varphi_t}g)'(z) = g'(z+t)$ and also $|(\mathcal{C}_{\varphi_t})'(g)|^2 = |g'(z+t)|^2$.

Now, for the part $\int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t}g)'(z)|^2 dA(z)$, we shall have,

$$\int_{\mathbb{U}} |g'(z+t)|^2 dA(z).$$

By changing variables, let $\omega = z+t$. Then $z = \omega - t$ and $dA(\omega) = dA(z)$. Thus,

$$\begin{aligned}
\|\mathcal{C}_{\varphi_t}g\|^2_{\mathcal{D}(\mathbb{U})} &= |\mathcal{C}_{\varphi_t}g(i)|^2 + \int_{\mathbb{U}} |(\mathcal{C}_{\varphi_t}g)'(z)|^2 dA(z) \\
&= |g(i+t)|^2 + \int_{\mathbb{U}} |g'(z+t)|^2 dA(\omega) \\
&= |g(i+t)|^2 + \int_{\mathbb{U}} |g'(\omega)|^2 dA(\omega)
\end{aligned}$$

is not an isometry since

$$|g(i+t)|^2 + \int_{\mathbb{U}} |g'(\omega)|^2 dA(\omega) \neq \|g\|_{\mathcal{D}(\mathbb{U})}.$$

□

Now, we argue as in the scaling class and consider $\mathcal{D}_\circ(\mathbb{U})$, and also define $\hat{\mathcal{C}}_{\varphi_t} : \mathcal{D}_\circ(\mathbb{U}) \to \mathcal{D}_\circ(\mathbb{U})$ by

$$\hat{\mathcal{C}}_{\varphi_t} g(z) = g(z+t) - g(i+t), \quad \forall g \in \mathcal{D}_\circ(\mathbb{U}). \tag{4.2}$$

Clearly $\hat{\mathcal{C}}_{\varphi_t}$ is well defined since $\hat{\mathcal{C}}_{\varphi_t} g(i) = g(i+t) - g(i+t) = 0$ and it also maps $\mathcal{D}_\circ(\mathbb{U})$ onto itself, as desired.
Next results gives the semigroup properties of $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ on $\mathcal{D}_\circ(\mathbb{U})$.

Proposition 4.3

Let $\hat{\mathcal{C}}_{\varphi_t}$ be as defined in (4.2). Then the following properties hold:

1. *The class $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ given in (4.2) forms a group of $\mathcal{D}_\circ(\mathbb{U})$.*
2. *$\hat{\mathcal{C}}_{\varphi_t}$ is a linear isometry on $\mathcal{D}_\circ(\mathbb{U})$.*
3. *$\hat{\mathcal{C}}_{\varphi_t}$ is strongly continuous on $\mathcal{D}_\circ(\mathbb{U})$.*
4. *The infinitesimal generator Γ of $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ is given by*

$$\Gamma g(\omega) = g'(\omega) - g'(i), \tag{4.3}$$

with dom$(\Gamma) = \{g \in \mathcal{D}_\circ(\mathbb{U}) : g'(\omega) \in \mathcal{D}(\mathbb{U})\}$.

PROOF. It is sufficient to show that $(\hat{\mathcal{C}}_{\varphi_t})_{t\geq 0}$ and $(\hat{\mathcal{C}}_{\varphi_t})_{t\leq 0}$ are semigroups on $\mathcal{D}_\circ(\mathbb{U})$. Indeed, for $g \in \mathcal{D}_\circ(\mathbb{U})$,

$$\begin{aligned} \hat{\mathcal{C}}_{\varphi_0}(z) &= g(z+0) - g(i+0) \\ &= g(z) - g(i) \\ &= g(z), \end{aligned}$$

so $\hat{\mathcal{C}}_{\varphi_0} = I$.

Also $\forall g \in \mathcal{D}(\mathbb{U})$, t, $s \geq 0$,

$$\begin{aligned}(\hat{\mathcal{C}}_{\varphi_t} \circ \hat{\mathcal{C}}_{\varphi_s})g(z) &= \hat{\mathcal{C}}_{\varphi_t}(\hat{\mathcal{C}}_{\varphi_s} g(z)) \\ &= \hat{\mathcal{C}}_{\varphi_t} g(z+t) - \hat{\mathcal{C}}_{\varphi_s} g(i+t) \\ &= g(z+t+s) - g(i+t+s) - (g(i+t+s) - g(i+t+s)) \\ &= g(z+(t+s)) - g(i+(t+s)) \\ &= \hat{\mathcal{C}}_{\varphi_{t+s}} g(z).\end{aligned}$$

Therefore, $\hat{\mathcal{C}}_{\varphi_t} \circ \hat{\mathcal{C}}_{\varphi_s} = \hat{\mathcal{C}}_{\varphi_{t+s}}$ and hence $(\hat{\mathcal{C}}_{\varphi_t})_{t \geq 0}$ form a semigroup on $\mathcal{D}_\circ(\mathbb{U})$.

We can also prove that $(\hat{\mathcal{C}}_{\varphi_t})_{t \leq 0}$ forms a semigroup on $\mathcal{D}_\circ(\mathbb{U})$ in a similar way. Therefore, $(\hat{\mathcal{C}}_{\varphi_t})_{t \in \mathbb{R}}$ forms a group under composition, as desired. This completes the proof of assertion 1.

For assertion 2, we first show that $\hat{\mathcal{C}}_{\varphi_t}$ is linear, $\forall\, f$, $g \in \mathcal{D}_\circ(\mathbb{U})$ and $\alpha, \beta \in \mathbb{F}$ we have;

$$\begin{aligned}\hat{\mathcal{C}}_{\varphi_t}(\alpha f + \beta g)(z) &= (\alpha f + \beta g)(z+t) - (\alpha f + \beta g)(i+t) \\ &= \alpha f(z+t) + \beta g(z+t) - \alpha f(i+t) - \beta g(i+t) \\ &= \alpha(f(z+t) - f(i+t)) + \beta(g(z+t) - g(i+t)) \\ &= \alpha \hat{\mathcal{C}}_{\varphi_t} f(z) + \beta \hat{\mathcal{C}}_{\varphi_t} g(z),\end{aligned}$$

as desired. We next prove that, $\hat{\mathcal{C}}_{\varphi_t}$ is an isometry. Now, $\forall g \in \mathcal{D}_\circ(\mathbb{U})$, we have;

$$\begin{aligned}
\|\hat{\mathcal{C}}_{\varphi_t} g\|^2_{\mathcal{D}\circ(\mathbb{U})} &= \int_{\mathbb{U}} |\hat{\mathcal{C}}_{\varphi_t}(g)'(\omega)|^2 dA(\omega) \\
&= \int_{\mathbb{U}} |(g(\omega+t) - g(i+t))'|^2 dA(\omega) \\
&= \int_{\mathbb{U}} |g'(\omega+t)|^2 dA(\omega).
\end{aligned}$$

Changing the variables, let $z = \omega + t$ then $\omega = z - t$ and $dA(z) = dA(\omega)$. Thus;

$$\begin{aligned}
\|\hat{\mathcal{C}}_{\varphi_t} g\|^2_{\mathcal{D}_\circ(\mathbb{U})} &= \int_{\mathbb{U}} |g'(\omega+t)|^2 dA(\omega) \\
&= \int_{\mathbb{U}} |g'(z)|^2 dA(z) \\
&= \|g\|^2_{\mathcal{D}\circ(\mathbb{U})}.
\end{aligned}$$

Strong continuity of the operator $\hat{\mathcal{C}}_{\varphi_t}$ on $\mathcal{D}_\circ(\mathbb{U})$ follows immediately from Theorem 3.1.

Next, we determine the infinitesimal generator Γ of $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$. If $g \in \mathrm{dom}(\Gamma)$ in $\mathcal{D}(\mathbb{U})$, then by definition we have,

$$\begin{aligned}
\Gamma g(\omega) &= \lim_{t\to 0^+} \frac{(g(\omega+t) - g(i+t)) - g(\omega)}{t} \\
&= \frac{\partial}{\partial t}(g(\omega+t) - g(i+t))\Big|_{t=0} \\
&= g'(\omega) - g'(i).
\end{aligned}$$

Hence $\mathrm{dom}(\Gamma) \subseteq \{g \in \mathcal{D}_\circ(\mathbb{U}) : g'(\omega) \in \mathcal{D}(\mathbb{U})\}$. Conversely, let

$g \in \mathcal{D}_\circ(\mathbb{U})$ such that $g'(\omega) \in \mathcal{D}(\mathbb{U})\ \forall \omega \in \mathbb{U}$, we have

$$\begin{aligned} \frac{\hat{\mathcal{C}}_{\varphi_t} g(\omega) - \hat{\mathcal{C}}_{\varphi_0} g(\omega)}{t-0} &= \frac{1}{t}\int_0^t \frac{\partial}{\partial s}(g(\omega) - g(i))ds \\ &= \frac{1}{t}\int_0^t \hat{\mathcal{C}}_{\varphi_s} G(\omega) ds \end{aligned}$$

Thus $\lim\limits_{t\to 0} \frac{\hat{\mathcal{C}}_{\varphi_t} g - g}{t} = \lim\limits_{t\to 0} \frac{1}{t}\int_0^t \mathcal{C}_{\varphi_s}(G)ds$ and strong continuity of $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ implies that $\frac{1}{t}\int_0^t \|\hat{\mathcal{C}}_{\varphi_t} G - G\| ds$ tends to zero as t tends to zero. Thus $\mathrm{dom}(\Gamma) \supseteq \{g \in \mathcal{D}_\circ(\mathbb{U}) : g'(\omega) \in \mathcal{D}(\mathbb{U})\}$. This gives the result. $\square$

4.2 Spectral properties

In this section we shall investigate the spectral properties of the infinitesimal generator, Γ. Specifically, we determine $\sigma(\Gamma)$, $\sigma_p(\Gamma)$ and $\rho(\Gamma)$ of Γ at a particular point in the resolvent set.

Proposition 4.4

Let Γ be the infinitesimal generator of the group $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ on the $\mathcal{D}_\circ(\mathbb{U})$, given in equation (4.3) *then, $\sigma_p(\Gamma) = \emptyset$ and $_\rho(\Gamma) \subseteq i\mathbb{R}$.*

PROOF. Let λ and g be an eigenvalue and eigenvector of Γ respectively. Then the equation $\Gamma g = \lambda g$ is equivalent to;

$$g'(\omega) - g'(i) = \lambda g(\omega). \tag{4.4}$$

To solve equation (4.4) we let $g'(i) = B$ to have

$$g'(\omega) - \lambda g(\omega) = B. \tag{4.5}$$

Equation (4.5) forms a first order differential equation and using an integrating factor technique we obtain the solution as,

$$g(\omega) = -\frac{B}{\lambda} + Ke^{\lambda\omega},$$

where, $B = g'(i)$ and the constant K is arbitrary.
By differentiation, $g'(\omega) = \lambda Ke^{\lambda\omega}$. It follows from lemma 3.4, assertion 2, $g' \notin L^2_a(\mathbb{U})$ and therefore $g \notin \mathcal{D}(\mathbb{U})$ for any λ implying that $\sigma_p(\Gamma) = \emptyset$.
Next we show that $\sigma(\Gamma) \subseteq i\mathbb{R}$.
Since each $(\hat{\mathcal{C}}_{\varphi_t})$ is an invertible isometry on $\mathcal{D}_\circ(\mathbb{U})$, (Theorem 2.3) its spectrum satisfies $\sigma(\hat{\mathcal{C}}_{\varphi_t}) \subseteq \partial\mathbb{D}$ and by Theorem 2.6 we have $e^{t\sigma(\Gamma)} \subseteq \sigma(\hat{\mathcal{C}}_{\varphi_t})$. Therefore $\sigma(\Gamma) \subseteq i\mathbb{R}$, which completes the proof. □

4.3 An integral operator on $\mathcal{D}_\circ(\mathbb{U})$

We now obtain an integral operator on $\mathcal{D}_\circ(\mathbb{U})$. Since $\sigma(\Gamma)$ is contained in $i\mathbb{R}$ we let a point $\lambda = 1$ in $\rho(\Gamma)$ and obtain $R(1, \Gamma)$, given by Theorem 2.2.

Theorem 4.5

Considering the infinitesimal generator, Γ *of* $(\hat{\mathcal{C}}_{\varphi_t})_{t\in\mathbb{R}}$ *on* $\mathcal{D}_\circ(\mathbb{U})$*, then these holds;*

(a) $R(1, \Gamma)$ *on* $\mathcal{D}_\circ(\mathbb{U})$ *is*

$$R(1, \Gamma)h(z) = \int_z^\infty \frac{1}{e^{(\omega - z)}}[h(\omega) - h(i + (\omega - z))]d\omega.$$

(b) $\sigma(R(1,\Gamma)) \subseteq \left\{\omega : |\omega - \frac{1}{2}| = \frac{1}{2}\right\}$.

(c) $\|R(1,\Gamma)\| \leq 1$.

(d) $r(R(1,\Gamma)) \leq 1$.

PROOF. Let us consider a point $\lambda = 1$. Since $\sigma(\Gamma) \subseteq i\mathbb{R}$, then $\lambda \in \rho(\Gamma)$. Therefore, applying the Laplace transform, (Theorem 2.2), we obtain $R(1,\Gamma)$ as;

$$\begin{aligned} R(1,\Gamma)h(z) &= \int_0^\infty e^{-t}\hat{C}_{\varphi_t}h dt \\ &= \int_0^\infty e^{-t}[h(z+t) - h(i+t)]dt \end{aligned}$$

Changing the variable and letting $\omega = z + t$, $t = \omega - z$ and $d\omega = dt$, $t = 0 \Rightarrow \omega = z, t = \infty \Rightarrow \omega = \infty$.

$$\begin{aligned} R(1,\Gamma)h(z) &= \int_z^\infty e^{-(w-z)}[h(\omega) - h(i + (\omega - z))]d\omega \\ &= \int_z^\infty \frac{1}{e^{(\omega-z)}}[h(\omega) - h(i + (\omega - z))]d\omega, \end{aligned}$$

which is the result.

Next, we estimate the spectrum of the resolvent $\sigma(R(\lambda,\Gamma))$ by applying Theorem 2.5 which asserts that $\sigma(R(\lambda,\Gamma)) \setminus \{0\} = (\lambda - \sigma(\Gamma))^{-1} = \left\{\frac{1}{\lambda-\mu} : \mu \in \sigma(\Gamma)\right\}$. Thus,

$$\sigma(R(\lambda,\Gamma)) \subseteq \left\{\frac{1}{\lambda - ir} : r \in \mathbb{R}\right\} \tag{4.6}$$

Let $\lambda = 1$ and substituting in (4.6) we get

$$\sigma(R(1,\Gamma)) \subseteq \left\{ \frac{1}{1-ir} : r \in \mathbb{R} \right\}.$$

Making the denominator rational and simplifying we have;

$$\sigma(R(1,\Gamma)) \subseteq \left\{ \frac{1+ir}{1+r^2} : r \in \mathbb{R} \right\}.$$

We let

$$\omega = \frac{1+ir}{1+r^2}$$

then we subtract $\frac{1}{2}$ and simplify we have,

$$\begin{aligned} \omega - \frac{1}{2} &= \frac{1-ir}{1+r^2} - \frac{1}{2} \\ &= \frac{-r+i}{2(1+r^2)}. \end{aligned}$$

Getting the magnitude on both sides we get,

$$\begin{aligned} \left|\omega - \frac{1}{2}\right|^2 &= \left|\frac{1-ir}{1+r^2} - \frac{1}{2}\right|^2 \\ &= \left|\frac{r^2+1}{4(r^2+1)}\right| \\ &= \frac{1}{4} \\ \omega - \frac{1}{2} &= \frac{1}{2}. \end{aligned}$$

Therefore, $\sigma(R(1,\Gamma)) \subseteq \left\{\omega : \left|\omega - \frac{1}{2}\right| = \frac{1}{2}\right\}$.

This proves (b).

Next, we apply Theorem 2.4 which proves that

$$\|R(\lambda, \Gamma)\| \leq \frac{1}{Re(\lambda)}.$$

Thus,

$$\|R(1, \Gamma)\| \leq 1.$$

Finally, we estimate $r(R(1, \Gamma))$.

Since a norm bounds an operator's spectral radius then, $r(R(1, \Gamma)) \leq \|R(1, \Gamma\|)$ which implies that

$$r(R(1, \Gamma)) \leq \|R(1, \Gamma)\| \leq 1.$$

Thus,

$$r(R(1, \Gamma)) \leq 1.$$

which completes proof of (d). $\square$

Chapter 5

Summary and recommendation

5.1 Summary

We evaluated all automorphism of $\mathbb{U}$ in this thesis, the automorphisms are classified into three distinctive classes. These classes are, translation, scaling and rotation classes (Theorem 2.10). In this study, our focus was solely on translation class and scaling class. Under the scaling class, the automorphism of the form (3.1) was extensively studied and we obtained the corresponding composition semigroup given by (3.5). We then investigated the spectral and semigroup properties of these composition semigroups. We constructed a class of integral operators on $\mathcal{D}_\circ(\mathbb{U})$ as resolvents to a group of isometries on $\mathcal{D}_\circ(\mathbb{U})$ which is strongly continuous. Moreover, we determined the norm and the spectra of the constructed class of integral operators and also, we determined the adjoint properties of the composition semigroup $\mathcal{D}_\circ(\mathbb{U})$. On the translation class, we also considered the automorphism of the form (3.2) and obtained the corresponding composition semigroup given by (4.1). In Proposition 4.3, we obtained Γ of the composition semigroup. We in addition confirmed that $\sigma_p(\Gamma)$ is empty and $\sigma(\Gamma) \subseteq i\mathbb{R}$ as outlined in Proposition 4.4. By applying (Theorem 2.2) which gave the resolvent of Γ as an integral operator acting on $\mathcal{D}_\circ(\mathbb{U})$ (Theorem 4.5). As a result, we used spectral mapping theorem (Theorem 2.5) to determine the spectrum of the integral operator. Finally, we applied Theorem 2.4 to obtain the integral operator's norm and the spectral radius.

5.2 Recommendations

Based on the study, we recommend further research in the following areas:

(i) The spectral analysis of the composition semigroup induced by the scaling class in now complete following this work and that of Oyugi [15]. However, the analysis of the composition semigroup induced by translation class remains open. Our study has partially obtained these properties. We therefore recommend a complete spectral analysis of the composition semigroup induced by the translation class.

(ii) From our study we have considered the composition semigroup on the subspace, $\mathcal{D}_\circ(\mathbb{U})$ of $\mathcal{D}(\mathbb{U})$. We therefore recommend a different approach should be used to study these operators on the Dirichlet space of the upper half plane, $\mathcal{D}(\mathbb{U})$, at large and obtain the spectral properties.

(iii) We constructed integral operators on $\mathcal{D}_{\circ}(\mathbb{U})$ and recommend an extension of this study to the setting of the weigted Dirichlet space.

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